

Covariant gravitational equations on a brane world with a Gauss-Bonnet term

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We present covariant gravitational equations to describe a four-dimensional brane world in the case with a Gauss-Bonnet term in a bulk spacetime, assuming that gravity is confined on the Z_2 symmetric brane. It contains some components of the five-dimensional Weyl curvature ($E_{\mu\nu}$) which describes all effects from the bulk spacetime just as in the case of the Randall-Sundrum second model. Applying this formalism to cosmology, we derive the generalized Friedmann equation and calculate the Weyl curvature term, which is directly obtained from a black hole solution.

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I. INTRODUCTION

A brane is now one of the most important ideas in particle physics [1]. It may provide us a new solution for the so-called hierarchy problem and a new mechanism for compactification of extra dimensions. Since the fundamental scale could be TeV in some models [2], the gravitational effect is not ignored even at a much lower energy scale than the Planck mass. For example, black hole formation in a next generation particle collider could be observed [3]. It should be also stressed that we could come across the first experimental evidence of quantum gravity. It may also change our view of the universe: we live in a four-dimensional (4D) hypersurface embedded in a higher-dimensional bulk spacetime [4]. For these reasons, the brane world scenarios have attracted a lot of attention.

Among the many brane models, those proposed by Randall and Sundrum (RS) are very important [5,6]. They are motivated by superstring or M theory: i.e., the orbifold compactification of higher-dimensional string theory by the dimensional reduction of 11-dimensional supergravity in $R^{10} \times S^1/Z_2$ [7]. The standard-model particles are confined in a 4D brane world while gravity accesses extra dimensions like a string or D-brane system. In their first model (RS I) [5], they proposed a mechanism to solve the hierarchy problem with two branes, whereas in their second model (RS II) [6], they considered a single brane with a positive tension, where 4D Newtonian gravity is recovered at low energies even if the extra dimension is not compact. This mechanism provides an alternative compactification of extra dimensions. However, those models may inevitably expect a singular spacetime just as in general relativity, although they are based on a string theory. In fact, Gibbons [8] and Maldacena and Nunez [9] showed a no-go theorem, which states that there are no nonsingular warped compactifications in a large class of supergravity theories including 11-dimensional supergravity, IIB, IIA, and massive IIA. One of the ways to evade this argument is adding the higher-curvature corrections to the bulk Lagrangian. The higher-curvature terms

naturally arise as a next leading order of the α' expansion of a superstring theory [10]. One may expect that they are described by the so-called Gauss-Bonnet combination, which is shown to be a ghost-free combination [11]. It also plays a fundamental role in Chern-Simon theories [12]. It was shown that the graviton zero mode is localized at low energies in the Gauss-Bonnet brane system as in the RS II model [13–16] (see Ref. [15] for a complete analysis) and that the correction of Newton's law becomes milder by including the Gauss-Bonnet term [17].

As for cosmology with a brane world, there has been many works over the last several years [18–20]. In particular, based on the RS II brane model, which is one of the most popular ones, some interesting properties such as “dark radiation” or quadratic density term in the Friedmann equation have been found, assuming a simple bulk metric [20]. Since gravity is confined on a brane, the induced metric describes gravity on a brane. Hence the geometrical reduction gives a covariant form of the basic equations for brane gravity [21–24]. Applying this formalism, we find the Friedmann equation easily.

As we discussed above, since including the Gauss-Bonnet term is important, such models have also been extensively studied [25–31]. Many authors so far have studied in the contexts of a resolution of initial singularity, inflation, and a self-tuning mechanism of cosmological constant and so on. In these analysis, a simple bulk metric is assumed just as in Ref. [20].

In order to understand those problems further, it may be convenient for us to extend the covariant gravitational equations on a brane to the case with the Gauss-Bonnet term. This is the purpose of the present paper. To find such equations, we first have to prove a consistency with a thin-wall ansatz. When we have a system with quadratic curvature terms in a bulk spacetime, we will be soon faced with an obstacle. In general, we expect terms such as $(\mathcal{E}_n K_{AB})^2$ in the field equations, where K_{AB} is the extrinsic curvature of a brane. If a brane is an infinitely thin singular wall, which could be described by the δ function, the extrinsic curvature must have a

jump at a brane. However, if $\mathcal{E}_n K_{AB}$ is proportional to the δ function, a term of $(\mathcal{E}_n K_{AB})^2$ makes trouble because it gives a square of the δ function. The reason for this breakdown is our thin-wall ansatz. We have to find other relevant junction conditions which may require information about the internal structure of a brane; that is, we have to discuss a thick-brane model. The basic equations may not be described only by a geometric reduction.

In the case with the Gauss-Bonnet term, however, the situation changes completely. The basic equations show a quasilinear property pointed out by Choquet-Bruhat [32] and Deruelle and Madore [33], which guarantees a thin-wall ansatz because it contains only linear terms of $\mathcal{E}_n K_{AB}$. Using this fact, some authors derived the generalized Friedmann equation with a simple bulk metric. With this fact, here we derive the covariant gravitational equations on a brane in the case with the Gauss-Bonnet term. The basic equations are described by 4D brane variables except for the 5D Weyl curvature tensor $E_{\mu\nu}$. Although our system is not closed because of the existence of $E_{\mu\nu}$, for a cosmological setting, we recover the generalized Friedmann equation which contains one integration constant and then gives a closed form just as in the case of Ref. [21]. This generalized Friedmann equation is the same as that obtained by the previous authors [29–31]. In this formulation, we need not to assume any functional form for the brane action. We can add any curvature terms in four dimensions, which may be induced by quantum effects of matter fields. These brane-induced gravity models were investigated mainly in the cosmological aspect [34–37].

In Sec. II, we derive the covariant gravitational equations on a brane, applying it to cosmological model in Sec. III. We obtain the generalized Friedmann equation, which is given by a cubic equation with respect to the Hubble parameter square H^2 . Conclusions and remarks follow in Sec. IV.

II. EFFECTIVE GRAVITATIONAL EQUATIONS

We consider a 5D bulk spacetime with a single 4D brane world, on which gravity is confined. We assume the 5D bulk spacetime (\mathcal{M}, g_{AB}) , whose coordinates are X^A ($A = 0, 1, 2, 3, 5$), is described by the Einstein-Gauss-Bonnet action

$$S_{\text{bulk}} = \int_{\mathcal{M}} d^5 X \sqrt{-g} \left[\frac{1}{2\kappa_5^2} (\mathcal{R} + \alpha \mathcal{L}_{GB}) + \mathcal{L}_m \right], \quad (2.1)$$

where

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{AB}\mathcal{R}^{AB} + \mathcal{R}_{ABCD}\mathcal{R}^{ABCD}. \quad (2.2)$$

Here κ_5^2 is the 5D gravitational constant, and \mathcal{R} , \mathcal{R}_{AB} , \mathcal{R}_{ABCD} , and \mathcal{L}_m are the 5D scalar curvature, Ricci tensor, Riemann curvature, and matter Lagrangian in the bulk, respectively. α is a coupling constant. The 4D brane world $(\mathcal{B}, h_{\mu\nu})$ is located at a hypersurface $[\Sigma(X^A) = 0]$ in the 5D bulk spacetime and the induced 4D metric $h_{\mu\nu}$ is defined by

$$h_{AB} = g_{AB} - n_A n_B, \quad (2.3)$$

where n_A is the spacelike unit-vector field normal to the brane hypersurface \mathcal{B} . The action is assumed to be given by the most generic action

$$S_{\text{brane}} = \int_{\mathcal{B}} d^4 x \sqrt{-h} [L_{\text{surface}} + L_{\text{brane}}(h_{\alpha\beta}, \psi)], \quad (2.4)$$

where x^μ ($\mu = 0, 1, 2, 3$) are the induced 4D coordinates on the brane,

$$L_{\text{surface}} = \frac{1}{\kappa_5^2} [K + 2\alpha(J - 2G^{\rho\sigma}K_{\rho\sigma})] \quad (2.5)$$

is the surface term [38–40], and $L_{\text{brane}}(h_{\alpha\beta}, \psi)$ is the effective 4D Lagrangian, which is given by a generic functional of the brane metric $h_{\alpha\beta}$ and matter fields ψ . Here $K_{\mu\nu}$ ($= h_\mu^A h_\nu^B \nabla_A n_B$), K , J , and $G^{\rho\sigma}$ in the surface term are the extrinsic curvature of \mathcal{B} , its trace, its cubic combination defined later, and the Einstein tensor of the induced metric $h_{\mu\nu}$, respectively.

The total action ($S = S_{\text{bulk}} + S_{\text{brane}}$) gives our basic equations as

$$\mathcal{G}_{AB} + \alpha \mathcal{H}_{AB} = \kappa_5^2 [\mathcal{T}_{AB} + \tau_{AB} \delta(\Sigma)], \quad (2.6)$$

where

$$\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2} g_{AB} \mathcal{R}, \quad (2.7)$$

$$\begin{aligned} \mathcal{H}_{AB} = & 2[\mathcal{R}\mathcal{R}_{AB} - 2\mathcal{R}_{AC}\mathcal{R}_B^C - 2\mathcal{R}^{CD}\mathcal{R}_{ACBD} \\ & + \mathcal{R}_A^{CDE}\mathcal{R}_{BCDE}] - \frac{1}{2} g_{AB} \mathcal{L}_{GB}, \end{aligned} \quad (2.8)$$

and

$$\mathcal{T}_{AB} \equiv -2 \frac{\delta \mathcal{L}_m}{\delta g^{AB}} + g_{AB} \mathcal{L}_m \quad (2.9)$$

is the energy-momentum tensor of bulk matter fields, while $\tau_{\mu\nu}$ is the “effective” energy-momentum tensor localized on the brane which is defined by

$$\tau_{\mu\nu} \equiv -2 \frac{\delta L_{\text{brane}}}{\delta h^{\mu\nu}} + h_{\mu\nu} L_{\text{brane}}. \quad (2.10)$$

The $\delta(\Sigma)$ denotes the localization of brane contributions. It is worth noting that $\tau_{\mu\nu}$ may include curvature contributions from induced gravity [34, 23]. In that term, we can also include “nonlocal” contributions such as a trace anomaly [36, 37], although those contributions are not directly derived from the effective Lagrangian L_{brane} .

The basic equations in the brane world are obtained by projecting the variables onto the brane world because we assume that the gravity on the brane is confined. We then project the 5D Riemann tensor onto the brane spacetime as

$$\mathcal{R}_{MNRS} h^M_A h^N_B h^R_C h^S_D = R_{ABCD} - K_{AC} K_{BD} + K_{AD} K_{BC}, \quad (2.11)$$

$$\mathcal{R}_{MNRS} h^M_A h^N_B h^R_C n^S = 2D_{[A} K_{B]C}, \quad (2.12)$$

$$\mathcal{R}_{MNRS} h^M_A h^R_C n^N n^S = -\mathfrak{L}_n K_{AC} + K_{AB} K^B_C, \quad (2.13)$$

where R_{ABCD} is the Riemann tensor of the induced metric h_{MN} , D_M is the covariant differentiation with respect to h_{MN} , and \mathfrak{L}_n denotes the Lie derivative in the n direction. The first equation is called the Gauss equation. Using this projection, the 5D Riemann curvature and its contractions (the Ricci tensor and scalar curvature) are described by the 4D variables on the brane with the normal n_M as

$$\begin{aligned} \mathcal{R}_{MNRS} = & R_{MNRS} - K_{MR} K_{NS} + K_{MS} K_{NR} - n_M D_R K_{NS} + n_M D_S K_{RN} + n_N D_R K_{SM} - n_N D_S K_{RM} \\ & - n_R D_M K_{NS} + n_R D_N K_{MS} + n_S D_M K_{NR} - n_S D_N K_{MR} + n_M n_R K_{NC} K^C_S - n_M n_S K_{NC} K^C_R \\ & - n_N n_R K_{MC} K^C_S + n_N n_S K_{MC} K^C_R - n_M n_R \mathfrak{L}_n K_{NS} + n_M n_S \mathfrak{L}_n K_{NR} + n_N n_R \mathfrak{L}_n K_{MS} - n_N n_S \mathfrak{L}_n K_{MR}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \mathcal{R}_{MN} = & R_{MN} - K K_{MN} + 2K_{MC} K^C_N + n_M (D_C K^C_N - D_N K) + n_N (D_C K^C_M - D_M K) \\ & + n_M n_N K_{CD} K^{CD} - \mathfrak{L}_n K_{MN} - n_M n_N h^{CD} \mathfrak{L}_n K_{CD}, \end{aligned} \quad (2.15)$$

$$\mathcal{R} = R - K^2 + 3K_{CD} K^{CD} - 2h^{CD} \mathfrak{L}_n K_{CD}. \quad (2.16)$$

As was shown in Refs. [32,33], the Einstein-Gauss-Bonnet equation is quasilinear, which means that apart from nonsingular terms given by the four-dimensional variables, it contains only linear terms of $\mathfrak{L}_n K_{AB}$ but no quadratic terms appear. In fact, inserting these relations into the basic equation (2.8), we find the effective equations on the brane as

$$\begin{aligned} M_{\mu\nu} - \frac{1}{2} M h_{\mu\nu} + K_{\mu\rho} K^\rho_\nu - h_{\mu\nu} K_{\alpha\beta} K^{\alpha\beta} - \mathfrak{L}_n K_{\mu\nu} + h_{\mu\nu} h^{\rho\sigma} \mathfrak{L}_n K_{\rho\sigma} \\ + 2\alpha (H_{\mu\nu} - M \mathfrak{L}_n K_{\mu\nu} + 2M_\mu{}^\rho \mathfrak{L}_n K_{\rho\nu} + 2M_\nu{}^\rho \mathfrak{L}_n K_{\rho\mu} + W_{\mu\nu}{}^{\rho\sigma} \mathfrak{L}_n K_{\rho\sigma}) = \kappa_5^2 [\mathcal{T}_{MN} h^M_\mu h^N_\nu + \tau_{\mu\nu} \delta(\Sigma)], \end{aligned} \quad (2.17)$$

$$N_\mu + 2\alpha (M N_\mu - 2M_\mu{}^\rho N_\rho + 2M^{\rho\sigma} N_{\mu\rho\sigma} - M_\mu{}^{\nu\rho\sigma} N_{\rho\sigma\nu}) = \kappa_5^2 \mathcal{T}_{MN} n^M n^N_\mu, \quad (2.18)$$

$$M + \alpha (M^2 - 4M_{\alpha\beta} M^{\alpha\beta} + M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}) = -2\kappa_5^2 \mathcal{T}_{MN} n^M n^N, \quad (2.19)$$

where

$$\begin{aligned} M_{\alpha\beta\gamma\delta} = & R_{\alpha\beta\gamma\delta} - K_{\alpha\gamma} K_{\beta\delta} + K_{\alpha\delta} K_{\beta\gamma}, \\ M_{\alpha\beta} = & h^{\rho\sigma} M_{\alpha\rho\beta\sigma} = R_{\alpha\beta} - K K_{\alpha\beta} + K_{\alpha\gamma} K^\gamma_\beta, \\ M = & h^{\alpha\beta} M_{\alpha\beta} = R - K^2 + K_{\alpha\beta} K^{\alpha\beta}, \end{aligned} \quad (2.20)$$

$$\begin{aligned} N_{\mu\nu\rho} = & D_\mu K_{\nu\rho} - D_\nu K_{\mu\rho}, \\ N_\mu = & h^{\rho\sigma} N_{\rho\mu\sigma} = D_\nu K_\mu{}^\nu - D_\mu K, \end{aligned} \quad (2.21)$$

$$\begin{aligned} H_{\mu\nu} = & M M_{\mu\nu} - 2(M_{\mu\rho} M^\rho_\nu + M^{\rho\sigma} M_{\mu\rho\nu\sigma}) + M_{\mu\rho\sigma\kappa} M_\nu{}^{\rho\sigma\kappa} + 2K_{\alpha\beta} K^{\alpha\beta} M_{\mu\nu} + M K_{\mu\rho} K^\rho_\nu \\ & - 2(K_{\mu\rho} K^\rho_\sigma M^\sigma_\nu + K_{\nu\rho} K^\rho_\sigma M^\sigma_\mu) - 2K^{\rho\kappa} K_\kappa{}^\sigma M_{\mu\rho\nu\sigma} - 2[N_\mu N_\nu - N^\rho (N_{\rho\nu\mu} + N_{\rho\mu\nu})] \\ & + N_{\rho\sigma\mu} N^{\rho\sigma}_\nu + 2N_{\mu\rho\sigma} N_\nu{}^{\rho\sigma} - \frac{1}{4} h_{\mu\nu} [M^2 - 4M_{\alpha\beta} M^{\alpha\beta} + M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}] \\ & + h_{\mu\nu} [-K_{\alpha\beta} K^{\alpha\beta} M + 2M_{\alpha\beta} K^{\alpha\gamma} K^\beta_\gamma + 2N_\alpha N^\alpha - N_{\alpha\beta\gamma} N^{\alpha\beta\gamma}], \end{aligned} \quad (2.22)$$

$$W_{\mu\nu}{}^{\rho\sigma} = M h_{\mu\nu} h^{\rho\sigma} - 2M_{\mu\nu} h^{\rho\sigma} - 2h_{\mu\nu} M^{\rho\sigma} + 2M_{\mu\alpha\nu\beta} h^{\alpha\rho} h^{\beta\sigma}. \quad (2.23)$$

Note that the linear terms of $\mathcal{L}_n K_{CD}$ appear only in Eq. (2.17) but not in Eqs. (2.18) and (2.19). This is consistent with our ansatz that the contribution from a brane is given by Eq. (2.10) because its contraction by n_A vanishes.

The singular behavior in 5D bulk spacetime appears in the 5D gravitational equations (2.6) as the δ function. Then, $\mathcal{L}_n K_{CD}$ has the δ -functional singularity to balance to the energy-momentum tensor of the brane world. Integrating Eq. (2.17) in the n direction, we obtain the generalized Israel's junction condition [41–43]

$$[K_{\mu\nu}]_{\pm} - h_{\mu\nu}[K]_{\pm} + 2\alpha(3[J_{\mu\nu}]_{\pm} - h_{\mu\nu}[J]_{\pm} - 2P_{\mu\rho\nu\sigma}[K^{\rho\sigma}]_{\pm}) = -\kappa_5^2 \tau_{\mu\nu}, \quad (2.24)$$

where

$$J_{\mu\nu} = \frac{1}{3}(2KK_{\mu\rho}K^{\rho}_{\nu} + K_{\rho\sigma}K^{\rho\sigma}K_{\mu\nu} - 2K_{\mu\rho}K^{\rho\sigma}K_{\sigma\nu} - K^2K_{\mu\nu}), \quad (2.25)$$

$$P_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + 2h_{\mu[\sigma}R_{\rho]\nu} + 2h_{\nu[\rho}R_{\sigma]\mu} + Rh_{\mu[\rho}h_{\sigma]\nu}. \quad (2.26)$$

We have introduced

$$[X]_{\pm} \equiv X^+ - X^-, \quad (2.27)$$

where X^{\pm} are X 's evaluated either on the $+$ or $-$ side of the brane and $P_{\mu\nu\rho\sigma}$ is the divergence-free part of the Riemann tensor: i.e.,

$$D_{\mu}P^{\mu}_{\nu\rho\sigma} = 0. \quad (2.28)$$

Because of the Z_2 symmetry, we have

$$K_{\mu\nu}^+ = -K_{\mu\nu}^-; \quad (2.29)$$

then, the extrinsic curvature of the brane is uniquely determined by the junction condition as

$$B_{\mu\nu} = -\frac{\kappa_5^2}{2}\tau_{\mu\nu}, \quad (2.30)$$

where

$$B_{\mu\nu} \equiv K_{\mu\nu} - Kh_{\mu\nu} + 2\alpha(3J_{\mu\nu} - Jh_{\mu\nu} - 2P_{\mu\rho\nu\sigma}K^{\rho\sigma}). \quad (2.31)$$

In what follows, we omit the indices \pm below for brevity.

The above quasilinearity guarantees the ansatz of an infinitely thin brane. The equations obtained for the induced metric are described by geometrical quantities and do not depend on the microphysics of the brane. This situation will be changed when we discuss other curvature-squared terms. On the other hand, if we include the Lovelock Lagrangian which is higher than the Gauss-Bonnet one but does not contain higher derivatives, we can assume that a brane is infinitely thin because it is also quasilinear and then extend the present approach.

In order to find the effective equations on the brane, we have to replace the terms of $\mathcal{L}_n K_{\mu\nu}$ in Eq. (2.17) with the 4D variables on the brane. The singular part in Eq. (2.17) has been evaluated by the junction condition. Hence we have to evaluate $\mathcal{L}_n K_{\mu\nu}$ either on the $+$ or $-$ side of the brane, which is nonsingular. From Eqs. (2.13), (2.15), and (2.16) with the decomposition of the Riemann tensor as

$$\mathcal{R}_{ABCD} = \frac{2}{3}(g_{A[C}\mathcal{R}_{D]B} - g_{B[C}\mathcal{R}_{D]A}) - \frac{1}{6}g_{A[C}g_{D]B}\mathcal{R} + \mathcal{C}_{ABCD}, \quad (2.32)$$

where \mathcal{C}_{MRNS} is the 5D Weyl curvature, we find

$$\begin{aligned} \mathcal{L}_n K_{\mu\nu} - \frac{1}{4}h_{\mu\nu}h^{\alpha\beta}\mathcal{L}_n K_{\alpha\beta} = & -\frac{3}{2}E_{\mu\nu} - \frac{1}{2}\left(M_{\mu\nu} - \frac{1}{4}h_{\mu\nu}M\right) \\ & + K_{\mu\rho}K^{\rho}_{\nu} - \frac{1}{4}h_{\mu\nu}K_{\rho\sigma}K^{\rho\sigma}, \end{aligned} \quad (2.33)$$

where

$$E_{\mu\nu} \equiv \mathcal{C}_{MRNS}n^M n^N h^R_{\mu} h^S_{\nu}. \quad (2.34)$$

However, because Eq. (2.33) is a trace-free equation, we cannot fix $\mathcal{L}_n K_{\mu\nu}$ by Eq. (2.33). We have to find $h^{\alpha\beta}\mathcal{L}_n K_{\alpha\beta}$ from other independent equations. We shall take a trace of our basic equation (2.6), finding

$$3\mathcal{R} + \alpha\mathcal{L}_{GB} = -2\kappa_5^2\mathcal{T}. \quad (2.35)$$

Inserting Eqs. (2.11)–(2.13) with Eq. (2.33) into Eq. (2.35), we find

$$h^{\alpha\beta}\mathcal{L}_n K_{\alpha\beta} = \frac{M}{2} + K_{\alpha\beta}K^{\alpha\beta} + \frac{\kappa_5^2}{3+\alpha M}\mathcal{T} + \frac{\alpha}{2(3+\alpha M)}I, \quad (2.36)$$

where

$$\begin{aligned} I = & M^2 - 8M_{\alpha\beta}M^{\alpha\beta} + M_{\alpha\beta\gamma\delta}M^{\alpha\beta\gamma\delta} \\ & - 8N_{\rho}N^{\rho} + 4N_{\rho\sigma\kappa}N^{\rho\sigma\kappa} - 12M_{\rho\sigma}E^{\rho\sigma}. \end{aligned} \quad (2.37)$$

From Eq. (2.33) with Eq. (2.36), we then find

$$\begin{aligned} \mathcal{L}_n K_{\mu\nu} = & -\frac{3}{2}E_{\mu\nu} - \frac{1}{2}\left(M_{\mu\nu} - \frac{1}{2}Mh_{\mu\nu}\right) + K_{\mu\rho}K^{\rho}_{\nu} \\ & + \frac{\kappa_5^2}{4(3+\alpha M)}\mathcal{T}h_{\mu\nu} + \frac{\alpha}{8(3+\alpha M)}Ih_{\mu\nu}. \end{aligned} \quad (2.38)$$

Inserting Eq. (2.38) into Eq. (2.17), we obtain the effective gravitational equations on the brane as

$$\begin{aligned}
& \frac{3}{2}(M_{\mu\nu} + E_{\mu\nu}) - \frac{1}{4}Mh_{\mu\nu} + \alpha[H_{\mu\nu}^{(1)} + H_{\mu\nu}^{(2)} + H_{\mu\nu}^{(3)}] \\
& - \frac{\alpha^2 I}{2(3 + \alpha M)} \left(M_{\mu\nu} - \frac{1}{4}Mh_{\mu\nu} \right) \\
& = \kappa_5^2 \left[T_{MN} h_{\mu}^M h_{\nu}^N - \frac{1}{4}h_{\mu\nu} T \right. \\
& \quad \left. + \frac{\alpha}{3 + \alpha M} \left(M_{\mu\nu} - \frac{1}{4}Mh_{\mu\nu} \right) T \right], \quad (2.39)
\end{aligned}$$

where

$$\begin{aligned}
H_{\mu\nu}^{(1)} &= 2M_{\mu\alpha\beta\gamma} M_{\nu}^{\alpha\beta\gamma} - 6M^{\rho\sigma} M_{\mu\rho\nu\sigma} + 4MM_{\mu\nu} - 8M_{\mu\rho} M_{\nu}^{\rho} \\
& - \frac{1}{8}h_{\mu\nu}(7M^2 - 24M_{\alpha\beta} M^{\alpha\beta} + 3M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}), \\
H_{\mu\nu}^{(2)} &= -6(M_{\mu\rho} E_{\nu}^{\rho} + M_{\nu\rho} E_{\mu}^{\rho} + M_{\mu\rho\nu\sigma} E^{\rho\sigma}) \\
& + \frac{9}{2}h_{\mu\nu} M_{\rho\sigma} E^{\rho\sigma} + 3ME_{\mu\nu}, \\
H_{\mu\nu}^{(3)} &= -4N_{\mu} N_{\nu} + 4N^{\rho}(N_{\rho\mu\nu} + N_{\rho\nu\mu}) + 2N_{\rho\sigma\mu} N^{\rho\sigma}_{\nu} \\
& + 4N_{\mu\rho\sigma} N_{\nu}^{\rho\sigma} + 3h_{\mu\nu} \left(N_{\alpha} N^{\alpha} - \frac{1}{2}N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right). \quad (2.40)
\end{aligned}$$

Equation (2.19) is automatically satisfied when we take a trace of Eq. (2.39), which means that it is not independent. We have then two basic equations (2.39) and (2.18). Equation (2.18) is rewritten as

$$\begin{aligned}
D_{\nu}[K_{\mu}^{\nu} - K\delta_{\mu}^{\nu} + 2\alpha(3J_{\mu}^{\nu} - J\delta_{\mu}^{\nu} - 2P_{\mu}^{\rho\nu\sigma} K_{\rho\sigma})] \\
= \kappa_5^2 T_{MN} h_{\mu}^M n^N, \quad (2.41)
\end{aligned}$$

which gives the constraint on the brane matter fields through the junction condition (2.30): i.e.,

$$D_{\nu}\tau_{\mu}^{\nu} = -2T_{MN} h_{\mu}^M n^N. \quad (2.42)$$

If there is no energy-momentum transfer from the bulk, we find the energy-momentum conservation of brane matter fields as

$$D_{\nu}\tau_{\mu}^{\nu} = 0. \quad (2.43)$$

In Eq. (2.39), we have so far three unknown variables T_{AB} , $E_{\mu\nu}$, and $K_{\mu\nu}$. The first two variables are described by bulk information, whereas the extrinsic curvature $K_{\mu\nu}$ is related to the brane “energy-momentum” tensor $\tau_{\mu\nu}$ as Eq. (2.30). Hence Eqs. (2.30) and (2.39) with the energy-momentum conservation (2.43) give the effective gravity theory on the brane.

It may be better to rewrite Eq. (2.39) to the Einstein-type equations with “correction” terms. From Eqs. (2.19) and (2.39), we find

$$\begin{aligned}
& G_{\mu\nu} + E_{\mu\nu} - KK_{\mu\nu} + K_{\mu\rho} K^{\rho}_{\nu} + \frac{1}{2}(K^2 - K_{\alpha\beta} K^{\alpha\beta})h_{\mu\nu} \\
& + \alpha(\hat{H}_{\mu\nu}^{(1)} + \hat{H}_{\mu\nu}^{(2)} + \hat{H}_{\mu\nu}^{(3)}) \\
& = \frac{2\kappa_5^2}{3} \left[\left[T_{MN} h_{\mu}^M h_{\nu}^N + \left(T_{MN} n^M n^N - \frac{1}{4}T^M_M \right) h_{\mu\nu} \right] \right. \\
& \quad \left. + \frac{\alpha}{3 + \alpha M} \left(M_{\mu\nu} - \frac{1}{4}Mh_{\mu\nu} \right) T_{MN} h^{MN} \right], \quad (2.44)
\end{aligned}$$

where

$$\begin{aligned}
\hat{H}_{\mu\nu}^{(1)} &= \frac{4}{3}(M_{\mu\alpha\beta\gamma} M_{\nu}^{\alpha\beta\gamma} - 3M^{\rho\sigma} M_{\mu\rho\nu\sigma} \\
& + 2MM_{\mu\nu} - 4M_{\mu\rho} M_{\nu}^{\rho}) \\
& - \frac{1}{12}h_{\mu\nu}(11M^2 - 40M_{\alpha\beta} M^{\alpha\beta} + 7M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}) \\
& - \frac{\alpha}{3(3 + \alpha M)} \left(M_{\mu\nu} - \frac{1}{4}Mh_{\mu\nu} \right) \\
& \times (M^2 - 8M_{\alpha\beta} M^{\alpha\beta} + M_{\alpha\beta\gamma\delta} M^{\alpha\beta\gamma\delta}), \\
\hat{H}_{\mu\nu}^{(2)} &= -4(M_{\mu\rho} E_{\nu}^{\rho} + M_{\nu\rho} E_{\mu}^{\rho} + M_{\mu\rho\nu\sigma} E^{\rho\sigma}) + 3h_{\mu\nu} M_{\rho\sigma} E^{\rho\sigma} \\
& + 2ME_{\mu\nu} + \frac{4\alpha}{3 + \alpha M} \left(M_{\mu\nu} - \frac{1}{4}Mh_{\mu\nu} \right) M_{\rho\sigma} E^{\rho\sigma}, \\
\hat{H}_{\mu\nu}^{(3)} &= \frac{8}{3} \left[-N_{\mu} N_{\nu} + N^{\rho}(N_{\rho\mu\nu} + N_{\rho\nu\mu}) \right. \\
& + \frac{1}{2}N_{\rho\sigma\mu} N^{\rho\sigma}_{\nu} + N_{\mu\rho\sigma} N_{\nu}^{\rho\sigma} \left. \right] \\
& + \left[2h_{\mu\nu} + \frac{8\alpha}{3(3 + \alpha M)} \left(M_{\mu\nu} - \frac{1}{4}Mh_{\mu\nu} \right) \right] \\
& \times \left(N_{\alpha} N^{\alpha} - \frac{1}{2}N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right). \quad (2.45)
\end{aligned}$$

As for the junction condition, we find

$$\begin{aligned}
& K_{\mu\nu} + \frac{2\alpha}{3} [9J_{\mu\nu} - 2Jh_{\mu\nu} - 2(3P_{\mu\rho\nu\sigma} + h_{\mu\nu} G_{\rho\sigma}) K^{\rho\sigma}] \\
& = -\frac{\kappa_5^2}{2} \left(\tau_{\mu\nu} - \frac{1}{3}\tau h_{\mu\nu} \right). \quad (2.46)
\end{aligned}$$

If $\alpha=0$, we find two equations

$$G_{\mu\nu} + E_{\mu\nu} - K K_{\mu\nu} + K_{\mu\rho} K^\rho_\nu + \frac{1}{2}(K^2 - K_{\alpha\beta} K^{\alpha\beta}) h_{\mu\nu} \\ = \frac{2\kappa_5^2}{3} \left[\mathcal{T}_{MN} h^M_\mu h^N_\nu + \left(\mathcal{T}_{MN} n^M n^N - \frac{1}{4} \mathcal{T}_M^M \right) h_{\mu\nu} \right], \quad (2.47)$$

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2} \left(\tau_{\mu\nu} - \frac{1}{3} \tau h_{\mu\nu} \right), \quad (2.48)$$

which are exactly the same as those found in Ref. [21], which gives the Einstein gravitational theory in the 4D brane world. However, if the Gauss-Bonnet term appears, the gravitational interaction on the brane will be modified in the effective theory.

The gravity on the brane is described by Eq. (2.39) with Eq. (2.30) or, equivalently, by Eq. (2.44) with Eq. (2.46). Just as the case of the RS II model, this system is not closed because of the appearance of terms with $E_{\mu\nu}$, which is some component of the 5D Weyl curvature. Although we have to solve a bulk spacetime as well as a brane world, we know that any contribution from a bulk spacetime to a brane world is described only by the tidal force $E_{\mu\nu}$.

Although the above form (2.39) or (2.44) is good enough to describe our basic equations, it is sometimes convenient to divide Eq. (2.39) into two parts: its trace and the trace-free equation. Introducing trace-free variables as

$$\tilde{M}_{\mu\nu} = M_{\mu\nu} - \frac{1}{4} M h_{\mu\nu}, \\ L_{\mu\nu\rho\sigma} = M_{\mu\nu\rho\sigma} + h_{\mu[\sigma} \tilde{M}_{\rho]\nu} + h_{\nu[\rho} \tilde{M}_{\sigma]\mu} - \frac{1}{6} M h_{\mu[\rho} h_{\sigma]\nu}, \quad (2.49)$$

we find

$$M + \alpha \left(\frac{1}{6} M^2 - 2 \tilde{M}_{\alpha\beta} \tilde{M}^{\alpha\beta} + L_{\alpha\beta\gamma\delta} L^{\alpha\beta\gamma\delta} \right) \\ = -2\kappa_5^2 \mathcal{T}_{MN} n^M n^N, \quad (2.50)$$

$$\frac{3}{2} (\tilde{M}_{\mu\nu} + E_{\mu\nu}) + \alpha [\bar{H}_{\mu\nu}^{(1)} + \bar{H}_{\mu\nu}^{(2)} + \bar{H}_{\mu\nu}^{(3)}] \\ = \kappa_5^2 \left[\mathcal{T}_{MN} h^M_\mu h^N_\nu - \frac{1}{4} h_{\mu\nu} \mathcal{T}_{MN} h^{MN} \right. \\ \left. + \frac{\alpha}{3 + \alpha M} \tilde{M}_{\mu\nu} \mathcal{T}_{MN} h^{MN} \right], \quad (2.51)$$

where

$$\bar{H}_{\mu\nu}^{(1)} = 2(L_{\mu\alpha\beta\gamma} L_\nu^{\alpha\beta\gamma} - \tilde{M}^{\alpha\beta} L_{\mu\alpha\nu\beta} - \tilde{M}_\mu^\alpha \tilde{M}_{\alpha\nu}) \\ - \frac{3 - \alpha M}{6(3 + \alpha M)} M \tilde{M}_{\mu\nu} + \frac{2\alpha}{3 + \alpha M} \tilde{M}_{\alpha\beta} \tilde{M}^{\alpha\beta} \tilde{M}_{\mu\nu} \\ - \frac{1}{2} h_{\mu\nu} (L_{\alpha\beta\gamma\delta} L^{\alpha\beta\gamma\delta} - \tilde{M}_{\alpha\beta} \tilde{M}^{\alpha\beta}),$$

$$\bar{H}_{\mu\nu}^{(2)} = -3(\tilde{M}_{\mu\rho} E^\rho_\nu + \tilde{M}_{\nu\rho} E^\rho_\mu + 2L_{\mu\rho\nu\sigma} E^{\rho\sigma}) \\ + \frac{3}{2} h_{\mu\nu} \tilde{M}_{\rho\sigma} E^{\rho\sigma} + \frac{1}{2} M E_{\mu\nu} \\ + \frac{6\alpha}{3 + \alpha M} \tilde{M}_{\rho\sigma} E^{\rho\sigma} \tilde{M}_{\mu\nu}, \\ \bar{H}_{\mu\nu}^{(3)} = -4N_\mu N_\nu + 4N^\rho (N_{\rho\mu\nu} + N_{\rho\nu\mu}) + 2N_{\rho\sigma\mu} N^{\rho\sigma}_\nu \\ + 4N_{\mu\rho\sigma} N_\nu^{\rho\sigma} + 3h_{\mu\nu} \left(N_\alpha N^\alpha - \frac{1}{2} N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right) \\ + \frac{4\alpha}{3 + \alpha M} \left(N_\alpha N^\alpha - \frac{1}{2} N_{\alpha\beta\gamma} N^{\alpha\beta\gamma} \right) \tilde{M}_{\mu\nu}. \quad (2.52)$$

The junction condition (2.30) is also decomposed into two parts

$$B \equiv B^\mu_\mu = -3K + \alpha \left(4\tilde{M}_{\rho\sigma} \tilde{K}^{\rho\sigma} - KM - \frac{1}{2} K^3 \right. \\ \left. + 2K \tilde{K}_{\rho\sigma} \tilde{K}^{\rho\sigma} - \frac{8}{3} \tilde{K}^\rho_\sigma \tilde{K}^\sigma_\kappa \tilde{K}^\kappa_\rho \right) = -\frac{\kappa_5^2}{2} \tau, \quad (2.53)$$

$$\tilde{B}_{\mu\nu} \equiv B_{\mu\nu} - \frac{1}{4} B h_{\mu\nu} \\ = \tilde{K}_{\mu\nu} - 12\alpha \tilde{J}_{\mu\nu} - \alpha \left[4L_{\mu\rho\nu\sigma} \tilde{K}^{\rho\sigma} + 2\tilde{K}_\mu^\rho \tilde{M}_{\rho\nu} \right. \\ \left. + 2\tilde{K}_\nu^\rho \tilde{M}_{\rho\mu} - h_{\mu\nu} \tilde{K}_{\rho\sigma} \tilde{M}^{\rho\sigma} - K \tilde{M}_{\mu\nu} - \frac{1}{3} M \tilde{K}_{\mu\nu} \right] \\ = -\frac{\kappa_5^2}{2} \left(\tau_{\mu\nu} - \frac{1}{4} \tau h_{\mu\nu} \right), \quad (2.54)$$

where

$$\tilde{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{4} K h_{\mu\nu}, \\ \tilde{J}_{\mu\nu} = J_{\mu\nu} - \frac{1}{4} J h_{\mu\nu} \\ = \frac{1}{3} \left[-2\tilde{K}_{\mu\rho} \tilde{K}^{\rho\sigma} \tilde{K}_{\sigma\nu} + \frac{1}{2} K \tilde{K}_{\mu\rho} \tilde{K}^\rho_\nu \right. \\ \left. + \tilde{K}_{\mu\nu} \left(\tilde{K}_{\rho\sigma} \tilde{K}^{\rho\sigma} - \frac{1}{8} K^2 \right) \right. \\ \left. + \frac{1}{2} h_{\mu\nu} \left(\tilde{K}^\alpha_\beta \tilde{K}^\beta_\gamma \tilde{K}^\gamma_\alpha - \frac{1}{4} K \tilde{K}_{\rho\sigma} \tilde{K}^{\rho\sigma} \right) \right]. \quad (2.55)$$

In this decomposition, our basic equations are Eqs. (2.50) and (2.51) with the junction conditions (2.53) and (2.54). This form may be better to describe some symmetric space-time such as the Friedmann-Robertson-Walker (FRW) universe, which we shall study next.

III. FRIEDMANN EQUATION

We apply the present reduction to the FRW cosmology. We assume spacetime as

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j, \quad (3.1)$$

where γ_{ij} denotes the metric of maximally symmetric three-dimensional space. This gives

$$\begin{aligned} R^0_0 &= 3(X+Y), \quad R^i_j = (Y+3X)\delta^i_j, \\ R^{0i}_{0j} &= (X+Y)\delta^i_j, \quad R^{ij}_{kl} = X(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k), \\ P^{0i}_{0j} &= X\delta^i_j, \quad P^{ij}_{kl} = (X+Y)(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k), \end{aligned} \quad (3.2)$$

where

$$X \equiv H^2 + \frac{k}{a^2}, \quad Y \equiv \dot{H} - \frac{k}{a^2}. \quad (3.3)$$

We assume that only a cosmological constant exists in the bulk: i.e.,

$$\kappa_5^2 \mathcal{T}_{MN} = -\Lambda g_{MN}. \quad (3.4)$$

From the symmetry of FRW spacetime, we can set

$$K^\mu_\nu = (K^0_0, \mathcal{K} \delta^i_j), \quad (3.5)$$

$$E^\mu_\nu = E^0_0 \left(1, -\frac{1}{3} \delta^i_j \right). \quad (3.6)$$

We note that E^μ_ν is trace free. We then write down the basic equations (2.50) and (2.51) as

$$2\bar{X} + \bar{Y} + 4\alpha\bar{X}(\bar{X} + \bar{Y}) = \frac{\Lambda}{3}, \quad (3.7)$$

$$3\bar{Y} + 2E^0_0 + \alpha(H^{(1)} + H^{(2)}) = -\frac{8\alpha\Lambda}{3 + \alpha M} \bar{Y}, \quad (3.8)$$

where

$$\begin{aligned} \bar{X} &= X - \mathcal{K}^2, \\ \bar{Y} &= Y - K^0_0 \mathcal{K} + \mathcal{K}^2, \end{aligned} \quad (3.9)$$

$$M = 6(2\bar{X} + \bar{Y}),$$

$$H^{(1)} \equiv -2\bar{Y} \left[2\bar{X} + 3\bar{Y} - \frac{\alpha}{3(3 + \alpha M)} (M^2 + 18\bar{Y}^2) \right],$$

$$H^{(2)} \equiv 4 \left(2\bar{X} - \bar{Y} + \frac{6\alpha}{3 + \alpha M} \bar{Y}^2 \right) E^0_0. \quad (3.10)$$

Here we have used the fact that $L_{\mu\nu\rho\sigma}$ vanishes and

$$\tilde{M}^\mu_\nu = \frac{3}{2} \bar{Y} \left(1, -\frac{1}{3} \delta^i_j \right). \quad (3.11)$$

We have also found that N_ρ and $N_{\alpha\beta\gamma}$ vanish as will be shown below. The nontrivial components of N_ρ and $N_{\alpha\beta\gamma}$ are calculated as

$$N_0 = -3\mathcal{N}, \quad N_{0i}{}^j = -N_{i0}{}^j = \mathcal{N} \delta_i^j, \quad (3.12)$$

where $\mathcal{N} \equiv \dot{\mathcal{K}} - H(K^0_0 - \mathcal{K})$, which gives the left-hand side (LHS) of Eq. (2.18) as

$$-3\mathcal{N}(1 + 4\alpha\bar{X}). \quad (3.13)$$

Since the RHS of Eq. (2.18) vanishes when only a cosmological constant appears in a bulk spacetime, we obtain

$$\mathcal{N} = \dot{\mathcal{K}} - H(K^0_0 - \mathcal{K}) = 0, \quad (3.14)$$

resulting that $N_\rho = N_{\alpha\beta\gamma} = 0$.

As for the junction condition (2.30), we first calculate $J_{\mu\nu}$ as

$$\begin{aligned} \bar{J}^\mu_\nu &= -\frac{1}{2} \mathcal{K}^2 (K^0_0 - \mathcal{K}) \left(1, -\frac{1}{3} \delta^i_j \right), \\ J &= -2\mathcal{K}^2 (3K^0_0 + \mathcal{K}), \end{aligned} \quad (3.15)$$

and then obtain

$$\begin{aligned} \bar{B}^\mu_\nu &= \frac{3}{4} [(K^0_0 - \mathcal{K})(1 + 4\alpha\bar{X} + 8\alpha\mathcal{K}^2) \\ &\quad + 8\alpha\mathcal{K}\bar{Y}] \left(1, -\frac{1}{3} \delta^i_j \right), \\ B &= -3(K^0_0 + 3\mathcal{K}) - 4\alpha[2\mathcal{K}^2(\mathcal{K} + 3K^0_0) \\ &\quad + 3(K^0_0 + 3\mathcal{K})\bar{X} + 6\alpha\mathcal{K}\bar{Y}]. \end{aligned} \quad (3.16)$$

The junction condition (2.30) gives two independent relations

$$\mathcal{K}(1 + 4\alpha\bar{X}) + \frac{8}{3} \alpha \mathcal{K}^3 = \frac{\kappa_5^2}{6} \tau^0_0, \quad (3.17)$$

$$(K^0_0 - \mathcal{K})[1 + 4\alpha(\bar{X} + 2\mathcal{K}^2)] + 8\alpha\mathcal{K}\bar{Y} = \frac{\kappa_5^2}{2} (\tau^1_1 - \tau^0_0). \quad (3.18)$$

We also find

$$\dot{\bar{X}} = 2H\bar{Y} \quad (3.19)$$

from Eq. (3.14) with $\dot{X} = 2HY$. We then recover the energy-momentum conservation law

$$\dot{\tau}^0_0 + 3H(\tau^0_0 - \tau^1_1) = 0 \quad (3.20)$$

from Eqs. (3.17) and (3.18).

Using Eq. (3.19) with Eq. (3.7), we obtain

$$\dot{\bar{X}} = 2H(1 + 4\alpha\bar{X})^{-1} \left[\frac{\Lambda}{3} - 2\bar{X}(1 + 2\alpha\bar{X}) \right], \quad (3.21)$$

which is easily integrated as

$$\bar{X}(1 + 2\alpha\bar{X}) = \frac{\Lambda}{6} + \frac{C}{a^4}, \quad (3.22)$$

where C is an integration constant.

If $\alpha = 0$, we have

$$\bar{X} = \bar{X}_0(a) \equiv \frac{\Lambda}{6} + \frac{C}{a^4}. \quad (3.23)$$

With Eq. (3.17) we find

$$X \equiv \bar{X}_0(a) + \mathcal{K}^2 = \frac{\kappa_5^4}{36} (\tau_0^0)^2 + \frac{\Lambda}{6} + \frac{C}{a^4}. \quad (3.24)$$

Setting $\tau_0^0 = -(\lambda + \rho)$, where λ is a positive tension of the brane and ρ is the energy density, we recover the well-known equation in brane cosmology, i.e.

$$H^2 + \frac{k}{a^2} = \frac{\Lambda_4}{3} + \frac{8\pi G}{3} \rho + \frac{\kappa_5^4}{36} \rho^2 + \frac{E_0^0}{3}, \quad (3.25)$$

where

$$\Lambda_4 = \frac{1}{2} \left(\Lambda + \frac{\kappa_5^4 \lambda^2}{6} \right), \quad 8\pi G = \frac{\kappa_5^4}{6} \lambda. \quad (3.26)$$

We have used $E_0^0 (= -3\bar{Y}/2) = 3C/a^4$, which is obtained from Eqs. (3.7), (3.8), and (3.23).

When $\alpha \neq 0$, solving the above quadratic equation, we obtain

$$\bar{X} = \bar{X}_{\pm}(a) \equiv \frac{1}{4\alpha} [-1 \pm \sqrt{1 + 8\alpha\bar{X}_0(a)}]. \quad (3.27)$$

Inserting $\mathcal{K}^2 = X - \bar{X}_{\pm}(a)$ into the square of Eq. (3.17), we find

$$[X - \bar{X}_{\pm}(a)] \left[1 + \frac{8}{3} \alpha X + \frac{4}{3} \alpha \bar{X}_{\pm}(a) \right]^2 = \frac{\kappa_5^4}{36} (\tau_0^0)^2. \quad (3.28)$$

This is a generalization of the Friedmann equation. If the brane contains only the matter field including a tension—i.e., $\tau_0^0 = -(\lambda + \rho)$ —it is a cubic equation with respect to $X = H^2 + k/a^2$ [29,30,42]. When we have the Einstein-Hilbert action on the brane such as an induced gravity [23,34], the generalized Friedmann equation becomes complicated, but it is still a cubic equation [31]. In the case with a trace anomaly, τ_0^0 contains not only λ , ρ , and X , but also Y and \dot{Y} . As a result, we have a very complicated equation.

The other independent equation (3.8) just gives the value of E_0^0 : i.e.,

$$E_0^0 = -\frac{3}{2} \bar{Y}_{\pm}(a) \left[1 + \frac{8\alpha\bar{Y}_{\pm}(a)}{3[1 + 4\alpha\bar{X}_{\pm}(a)]} \right], \quad (3.29)$$

where

$$\begin{aligned} \bar{Y}_{\pm}(a) &= [1 + 4\alpha\bar{X}_{\pm}(a)]^{-1} \left[\frac{\Lambda}{3} - 2\bar{X}_{\pm}(a)[1 + 2\alpha\bar{X}_{\pm}(a)] \right] \\ &= -2C[1 + 4\alpha\bar{X}_{\pm}(a)]^{-1} a^{-4}. \end{aligned} \quad (3.30)$$

This is the advantage in our description of the basic equations (2.50) and (2.51). The former gives the generalized Friedmann equation, while the latter is an algebraic equation for E_0^0 .

Because we have the Birkhoff-type theorem [29,44], the bulk spacetime is described by a “black hole” solution [45,46], whose metric is

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Sigma_k^2, \quad (3.31)$$

where

$$f(r) = k + \frac{r^2}{4\alpha} \left(1 \mp \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{16\alpha\mu}{3r^4}} \right). \quad (3.32)$$

A mass of a “black hole” is given by $M = \Omega_k \mu / \kappa_5^2$, where Ω_k is the volume of three-dimensional space with a unit radius. Calculating the curvature tensor, we find

$$\begin{aligned} E_0^0 &= C^{01}_{01} \\ &= \pm \frac{\mu}{r^4} \left(1 + \frac{4\alpha\Lambda}{3} + \frac{16\alpha\mu}{3r^4} \right)^{-3/2} \left(1 + \frac{4\alpha\Lambda}{3} + \frac{16\alpha\mu}{9r^4} \right). \end{aligned} \quad (3.33)$$

Using Eq. (3.27), we find that Eq. (3.29) with Eq. (3.30) is the same as Eq. (3.33) with $\mu = 3C/2$ and $r = a$. Hence an integration constant C corresponds to a mass of a “black hole” just as in the RS II model. However, “dark radiation” E_0^0 is not simply a radiation term but it depends complicatedly on a . The signature \pm in Eq. (3.33) corresponds to \pm of \bar{X}_{\pm} .

Here we shortly summarize a global structure of the “black hole” solution (3.31). It has a singularity at $r = 0$ if $\mu \neq 0$ as

$$\mathcal{R}_{ABCD} \mathcal{R}^{ABCD} \approx \frac{4\mu}{ar^4}. \quad (3.34)$$

If $1 + \frac{4}{3} \alpha\Lambda > 0$, the upper-sign solution of Eq. (3.32) has an event horizon if $k \leq 0$ or $k = 1$ and $\mu > 3\alpha$, while for the lower-sign case, a singularity becomes naked unless $k = -1$ and $\mu < 3\alpha$. For the case of $1 + \frac{4}{3} \alpha\Lambda < 0$, r is bounded from above as $r \leq r_{\max}$, where

$$r_{\max} = \left(\frac{16\alpha\mu}{3|1 + \frac{4}{3} \alpha\Lambda|} \right)^{1/4}. \quad (3.35)$$

Although the equation $f(r)=0$ has a positive root for some restricted conditions, there appears another singularity at $r=r_{\max}$. The curvature-invariant diverges there as

$$\mathcal{R}_{ABCD}\mathcal{R}^{ABCD} \approx \frac{\mu}{12\alpha r_{\max}(r_{\max}-r)^3}. \quad (3.36)$$

Since this singularity is time like and then naked, the above root does not mean an event horizon. There is no regular asymptotic region.

Another important property of this solution is about stability. First, Boulware and Deser claimed that the lower-branch solution in Eq. (3.32) is unstable [45]; however, it turns out that the solutions in both branches are stable [47].

With these basic equations, several authors analyzed the dynamics of the universe [29–31]. In this paper, assuming the induced gravity model [23,34], we first find a condition for a Minkowski brane. In the induced gravity model [23], we have

$$\tau_0^0 = -(\lambda + \rho) + 3\mu_I^2 X, \quad (3.37)$$

where λ is a positive tension of a brane, ρ is the energy density on a brane, and μ_I is a mass scale in the induced gravity, which is expected to be the Planck mass. If we set $\mu_I=0$, we find the model without an induced gravity action on the brane. In the Minkowski brane, $X=Y=0$ and $\rho=0$. From these conditions with Eqs. (3.18) and (3.30), we show that $C=0$. The real value condition for \bar{X} requires

$$1 + \frac{4}{3}\alpha\Lambda > 0. \quad (3.38)$$

Inserting the conditions for the Minkowski brane with $C=0$ into Eq. (3.28), we find

$$\alpha\kappa_5^4\lambda^2 = 1 - 4\alpha\Lambda \mp \left(1 + \frac{4}{3}\alpha\Lambda\right)^{3/2}. \quad (3.39)$$

This is a tuning condition for a zero cosmological constant on the brane. For the upper branch, when we take a limit of $\alpha \rightarrow 0$, we recover $\Lambda + \kappa_5^4\lambda^2/6 = 0$, which is the fine-tuning condition for the RS II model. Note that such a limit does not exist for the lower branch, although we have the Minkowski brane in this branch. The condition (3.38) gives the possible range for λ : that is,

$$0 \leq \alpha\kappa_5^4\lambda^2 < 4 \quad \begin{cases} \text{for } -3 < 4\alpha\Lambda \leq 0 & \text{in the upper branch,} \\ \text{for } -3 < 4\alpha\Lambda \leq 9 & \text{in the lower branch.} \end{cases} \quad (3.40)$$

If Λ vanishes, we obtain the Minkowski brane for the lower branch with $\alpha\kappa_5^4\lambda^2=2$. A de Sitter brane (or anti de-Sitter brane) is obtained if λ is larger (or smaller) than that given by Eq. (3.39).

Finally, we show an asymptotic Friedmann equation by perturbing the Minkowski brane spacetime. Here we do not impose the tuning condition (3.39). Setting X , ρ , and C/a^4 as

small variables and expanding Eq. (3.28) up to those first-order terms, we find the conventional Friedmann equation with dark radiation as

$$H^2 + \frac{k}{a^2} = \frac{\Lambda_4^{(\pm)}}{3} + \frac{8\pi G_N^{(\pm)}}{3}\rho + \frac{C^{(\pm)}}{a^4}, \quad (3.41)$$

where

$$\Lambda_4^{(\pm)} = \frac{\alpha\kappa_5^4\lambda^2 - 1 + 4\alpha\Lambda \pm \left(1 + \frac{4}{3}\alpha\Lambda\right)^{3/2}}{12\alpha\left(1 - \frac{4}{9}\alpha\Lambda + \frac{1}{6}\kappa_5^4\lambda\mu_I^2\right)}, \quad (3.42)$$

$$8\pi G_N^{(\pm)} = \frac{\kappa_5^4}{6\left(1 - \frac{4}{9}\alpha\Lambda + \frac{1}{6}\kappa_5^4\lambda\mu_I^2\right)}(\lambda - \mu_I^2\Lambda_4^{(\pm)}), \quad (3.43)$$

$$C^{(\pm)} = \frac{C}{3\left(1 - \frac{4}{9}\alpha\Lambda + \frac{1}{6}\kappa_5^4\lambda\mu_I^2\right)} \times \left(2 \pm \sqrt{1 + \frac{4}{3}\alpha\Lambda + \frac{8}{3}\alpha\Lambda_4^{(\pm)}}\right). \quad (3.44)$$

Hence, in both branches, we have recovered the conventional Friedmann universe in an asymptotic form. The early stage of the universe may depend on the parameters as discussed in Refs. [29–31].

If $1 + \frac{4}{3}\alpha\Lambda < 0$, the scale factor of the universe cannot be infinitely large. There is an upper bound as $a < a_{\max} = r_{\max}$. No Minkowski brane exists. If the scale factor approaches this value, the Weyl curvature (3.33) diverges, where a singularity appears in a bulk black hole spacetime. Hence our universe evolves into a singularity although a scale factor is finite. Even if the universe does not approach this singularity, the universe will get into trouble because it is a naked singularity.

IV. CONCLUDING REMARKS

We have derived the covariant gravitational equations of a brane world model with the Gauss-Bonnet curvature-squared term in a bulk spacetime. Although the obtained equations are very complicated, any effects from a bulk spacetime to a brane world are described only by the Weyl curvature ($E_{\mu\nu}$). The basic equations are not given in a closed form because of this term.

Giving the energy-momentum tensor of the brane, which is shown to be conserved, the extrinsic curvature ($K_{\mu\nu}$) of a brane satisfies a cubic matrix equation. Since it is not explicitly given by the energy-momentum tensor, we have to solve a couple of equations for the induced metric and the extrinsic curvature. If the brane action includes the induced gravity term, which may be expected from quantum effects of matter fields on the brane, we have to replace the energy-

momentum tensor with its generalization just as in Ref. [23].

We have then applied the present formalism to cosmology. Assuming the FRW spacetime for a brane world, we have rederived the generalized Friedmann equation. The obtained equation has one integration constant, just as in the RS II model, which is proportional to mass of a 5D black hole solution. Hence the cosmological model has only one unknown parameter. The system is described in a closed form.

Although we consider the Gauss-Bonnet term which appears as the lowest-order correction in the α' expansion of superstring theory, we also expect higher-curvature correction terms which are not taken into account in the present system. These terms would become important in the region with large curvature and modify the above story around singularities. In this sense it should be noted that the present approach can be applied not only to a brane model with the Gauss-Bonnet term in arbitrary dimensions but also to that with any Lovelock terms because of their quasilinearity [48,33]. Another extension is inclusion of a dilaton field. In a

realistic string theory, we have a dilaton field which couples to the Gauss-Bonnet term as well. It will change the dynamics of a brane world too. Such extensions are in progress. Analyzing those models, we hope that some fundamental cosmological problems such as a big-bang singularity or a cosmological constant will be solved.

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